Chapter 5

Weak and Weak* Topologies

5.1 The Weak Topology

In this chapter, we will study topologies on Banach spaces which are weaker (*i.e.* coarser) than the norm topology.

Definition 5.1.1 *Let V be a Banach space. The* **weak topology** *on V is the coarsest (i.e. smallest) topology such that every element of V^* is continuous. Open (respectively, closed) sets in the weak topology will be called weakly open (respectively, weakly closed) sets.* ■

We have already encountered the notion of the weak topology on a given set such that a family of functions is continuous (cf. Definition 1.2.10). The weakly open sets are precisely the class of all arbitrary unions of finite intersections of sets of the form $f^{-1}(U)$ where $f \in V^*$ and U is an open set in $\mathbb{R}$ (or $\mathbb{C}$, in the case of complex Banach spaces).

A basic neighbourhood system for the weak topology is, therefore, the collection of sets of the form

$$U = \{x \in V \mid |f_i(x - x_0)| < \varepsilon \text{ for all } i \in I\}$$

where $x_0 \in V$, $\varepsilon > 0$, I is a finite indexing set and $f_i \in V^*$ for all $i \in I$. The set U described above forms a weakly open neighbourhood of the point $x_0 \in V$ (cf. the discussion following Definition 1.2.10).

Proposition 5.1.1 *The weak topology is Hausdorff.*

Proof: Let x and y be distinct points in V. Then, since V^* separates points in V (cf. Remark 3.1.1), there exists $f \in V^*$ such that $f(x) \neq f(y)$. Choose disjoint open neighbourhoods U of $f(x)$ and V

of $f(y)$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint weakly open neighbourhoods of x and y respectively. This completes the proof. ■

Notation: Given a sequence $\{x_n\}$ in V, we write $x_n \to x$ if the sequence converges to $x \in V$ in the norm topology, *i.e.* if $\|x_n - x\| \to 0$ as $n \to \infty$. If the sequence converges to x in the weak topology, we write $x_n \rightharpoonup x$.

Proposition 5.1.2 *Let V be a Banach space and let $\{x_n\}$ be a sequence in V.*
(i) *$x_n \rightharpoonup x$ in V if, and only if, $f(x_n) \to f(x)$ for all $f \in V^*$.*
(ii) *If $x_n \to x$ in V, then $x_n \rightharpoonup x$.*
(iii) *If $x_n \rightharpoonup x$ in V, then $\{\|x_n\|\}$ is bounded and*

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|.$$

(iv) *If $x_n \rightharpoonup x$ in V and $f_n \to f$ in V^*, then $f_n(x_n) \to f(x)$.*

Proof: (i) This is a direct consequence of the definition of the weak topology.

(ii) Let $f \in V^*$ be an arbitrary element. Then

$$|f(x_n) - f(x)| \leq \|f\| \, \|x_n - x\| \to 0.$$

The result now follows from (i).

(iii) This follows from (i) and the Banach-Steinhaus theorem (cf. Corollary 4.2.2 applied to the sequence $\{J_{x_n}\}$ in V^{**}).

(iv) We have

$$\begin{aligned} |f_n(x_n) - f(x)| &\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \\ &\leq \|f_n - f\| \, \|x_n\| + |f(x_n) - f(x)|. \end{aligned}$$

The first term on the right-hand side tends to zero since $\|x_n\|$ is bounded (by (iii)) and $\|f_n - f\| \to 0$. The second term also tends to zero (by (i)). This completes the proof. ■

Example 5.1.1 Consider ℓ_2 the space of all square summable real sequences. We can identify ℓ_2^* with ℓ_2 (cf. Example 3.1.1). Consider the sequence $\{\mathbf{e}_n\} \in \ell_2$ (cf. Example 3.1.1). If $x \in \ell_2 = \ell_2^*$, with $x = (x_i)$, we have that

$$< x, \mathbf{e}_n >_{\ell_2^*, \ell_2} = x_n$$

which tends to zero since $\sum_{i=1}^{\infty} |x_i|^2 < \infty$. Thus, the sequence $\{\mathbf{e}_n\}$ converges weakly to $\mathbf{0}$ in ℓ_2. Notice that this sequence has no subsequence which converges in the norm topology since

$$\|\mathbf{e}_n - \mathbf{e}_m\|_2 = \sqrt{2}$$

for all $n \neq m$. Thus, while norm convergence implies weak convergence, nothing can be said about the reverse implication.

We can similarly prove that for all $1 < p < \infty$, the sequence $\{\mathbf{e}_n\}$ converges weakly to zero in ℓ_p. ■

Proposition 5.1.3 *If V is a finite dimensional space, then the norm and weak topologies coincide.*

Proof: Since the weak topology is coarser than the norm topology, every weakly open set is also open in the norm topology. We thus have to prove the converse. Let U be open in the norm topology and let $x_0 \in U$. There exists $r > 0$ such that the open ball $B(x_0; r) \subset U$. Let $\dim(V) = n$ and let $\{v_1, \cdots, v_n\}$ be a basis for V such that, without loss of generality, $\|v_i\| = 1$ for all $1 \leq i \leq n$. If $x \in V$, then $x = \sum_{i=1}^{n} x_i v_i$ and define f_i to be the i-th coordinate projection, *i.e.* $f_i(x) = x_i$. Then

$$\|x - x_0\| = \left\|\sum_{i=1}^{n} f_i(x - x_0) v_i\right\| \leq \sum_{i=1}^{n} |f_i(x - x_0)|.$$

Define

$$W = \left\{x \in V \mid |f_i(x - x_0)| < \frac{r}{n},\ 1 \leq i \leq n\right\}.$$

Then W is a weakly open neighbourhood of x_0 and it is clear from the above computations that $W \subset B(x_0; r) \subset U$. Thus U is open in the weak topology as well and this completes the proof. ■

Thus, in a finite dimensional space the weak and norm open (respectively, closed) sets are the same. However, in infinite dimensional spaces, the weak topology is strictly coarser than the norm topology. We will presently see examples of norm closed (respectively, open) sets which are not closed (respectively, open) in the weak topology (cf. Examples 5.1.2 and 5.1.3 below). However, for convex sets, the situation is different.

Proposition 5.1.4 *Let C be a convex and (norm) closed subset of a Banach space V. Then C is also weakly closed. (The converse is always true, even without the convexity hypothesis).*

Proof: We will assume that V is a real Banach space, for simplicity. Let C be a closed and convex set in V and let $x_0 \notin C$. Then, by the Hahn-Banach theorem, there exists $f \in V^*$ and $\alpha \in \mathbb{R}$ such that $f(x_0) < \alpha < f(x)$ for all $x \in C$. Then the set

$$U = \{x \in V \mid f(x) < \alpha\}$$

is a weakly open neighbourhood of x_0 which does not meet C. Thus the complement of C is weakly open and so C is weakly closed. ■

Definition 5.1.2 *Let X be a topological space and let $f : X \to \mathbb{R}$ be a given function. We say that f is* **lower semi-continuous** *if, for every $\alpha \in \mathbb{R}$, the set*

$$f^{-1}((-\infty, \alpha]) = \{x \in X \mid f(x) \leq \alpha\}$$

is closed in X. ■

Clearly, every continuous map is lower semi-continuous. If $x_n \to x$ in X, and if $f : X \to \mathbb{R}$ is lower semi-continuous, then

$$f(x) \leq \liminf_{n \to \infty} f(x_n). \tag{5.1.1}$$

For, if $\alpha = \liminf_{n\to\infty} f(x_n)$, then, given any $\varepsilon > 0$, there exists a subsequence x_{n_k} such that $f(x_{n_k}) \leq \alpha + \varepsilon$ for all k. Since $f^{-1}((-\infty, \alpha + \varepsilon])$ is closed, it follows that $f(x) \leq \alpha + \varepsilon$ and since $\varepsilon > 0$ was arbitrarily chosen, (5.1.1) follows.

Corollary 5.1.1 *Let V be a Banach space and let $\varphi : V \to \mathbb{R}$ be* convex *and lower semi-continuous (with respect to the norm topology). Then φ is also lower semi-continuous with respect to the weak topology. In particular, the map $x \mapsto \|x\|$, being continuous, is also lower semi-continuous with respect to the weak topology and, if $x_n \rightharpoonup x$ in V, we have*

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|. \tag{5.1.2}$$

Proof: For every $\alpha \in \mathbb{R}$, the set $\varphi^{-1}((-\infty, \alpha])$ is closed (in the norm topology) and is convex. Hence it is weakly closed. This completes the proof. ■

Notation: Let V be Banach space. We will use the following notations.

$$\begin{aligned} D &= \{x \in V \mid \|x\| < 1\} \text{ (open unit ball).} \\ B &= \{x \in V \mid \|x\| \leq 1\} \text{ (closed unit ball).} \\ S &= \{x \in V \mid \|x\| = 1\} \text{ (unit sphere).} \end{aligned}$$

Example 5.1.2 Let V be an infinite dimensional Banach space. Let S be the unit sphere in V. Then S is never weakly closed, though it is closed in the norm topology. To see this, let $x_0 \in V$ such that $\|x_0\| < 1$. Consider any weakly open neighbourhood U of x_0 of the form

$$U = \{x \in V \mid |f_i(x - x_0)| < \varepsilon, 1 \leq i \leq n\}$$

where $\varepsilon > 0$ and $f_i \in V^*$ for $1 \leq i \leq n$. Consider the map $\mathcal{A} : V \to \mathbb{R}^n$ defined by

$$\mathcal{A}(x) = (f_1(x), \cdots, f_n(x)).$$

This map cannot be injective (otherwise, we will have $\dim(V) \leq n$, which is a contradiction). Thus, there exists $y_0 \neq \mathbf{0}$, such that $f_i(y_0) = 0$ for all $1 \leq i \leq n$. Then $x_0 + ty_0 \in U$ for all $t \in \mathbb{R}$. Set $g(t) = \|x_0 + ty_0\|$. Then $g(0) < 1$ while $g(t) \to +\infty$ as $t \to +\infty$. Hence, there exists t_0 such that $g(t_0) = 1$. Thus, $x_0 + t_0 y_0 \in U \cap S$. We have thus proved that every weakly open neighbourhood of every point in the open unit ball, D, intersects the unit sphere, S. Hence the closed unit ball, B, must lie in the weak closure of S. But B being closed (in the norm topology) and convex, is itself weakly closed. Thus the weak closure of the unit sphere, S, is the closed unit ball, B. Thus, S is not weakly closed. ■

Example 5.1.3 Let V be an infinite dimensional Banach space. Then the open unit ball D is not weakly open. As seen in the preceding example, every weakly open neighbourhood of a point $x_0 \in D$ contains an affine subspace of the form $\{x_0 + ty_0 \mid t \in \mathbb{R}\}$ where y_0 is chosen as before. Thus D cannot contain a weakly open neighbourhood of any of its points and hence D cannot be weakly open. ■

Thus, in an infinite dimensional Banach space, the weak topology is strictly coarser than the norm topology.

Proposition 5.1.5 (**Schur's Lemma**) *In the space ℓ_1, a sequence is convergent in the weak topology if, and only if, it converges in the norm topology.*

Proof: By Proposition 5.1.2, every sequence which converges in norm, also converges weakly. Conversely, let $\{x_n\}$ be a weakly convergent sequence. Without loss of generality, assume that $x_n \rightharpoonup \mathbf{0}$. Let

$$x_n = (x_n^1, x_n^2, \cdots, x_n^k, \cdots).$$

Consider the functional f_i which is the projection to the i-th coordinate. Then, since the sequence weakly converges to zero, it follows that $f_i(x_n) \to 0$, *i.e.*

$$\lim_{n\to\infty} x_n^i = 0$$

for very positive integer i. Assume, if possible, that $\{x_n\}$ does not converge to zero in norm. Then, there exist $\varepsilon > 0$ such that, for infinitely many n,

$$\sum_{k=1}^{\infty} |x_n^k| \geq \varepsilon.$$

Thus, working with a suitable subsequence if necessary, we may assume that this is true for all n.

Set $n_0 = m_0 = 1$. Define, for $k \geq 1$, n_k and m_k inductively as follows.

- n_k is the smallest integer greater than n_{k-1} such that

$$\sum_{j=1}^{m_{k-1}} |x_{n_k}^j| < \frac{\varepsilon}{5}.$$

(This is possible since we know that each coordinate sequence tends to zero.)

- Now choose m_k to be the smallest integer greater than m_{k-1} such that

$$\sum_{j=m_k+1}^{\infty} |x_{n_k}^j| < \frac{\varepsilon}{5}.$$

(This is possible since the sequence $(x_{n_k}) \in \ell_1$.)

Now define $y = (y^j) \in \ell_\infty = \ell_1^*$ as follows:

For $m_{k-1}+1 \leq j \leq m_k$,

$$y^j = \begin{cases} 0 & \text{if } x^j_{n_k} = 0, \\ \frac{|x^j_{n_k}|}{x^j_{n_k}} & \text{otherwise.} \end{cases}$$

By varying k over all positive integers, y^j will be defined for all positive integers j. Clearly $\|y\|_\infty = 1$. Also

$$\left| \sum_{j=1}^{\infty} (x^j_{n_k} y^j - |x^j_{n_k}|) \right| \leq 2 \sum_{j=1}^{m_{k-1}} |x^j_{n_k}| + 2 \sum_{j=m_k+1}^{\infty} |x^j_{n_k}| \leq \frac{4\varepsilon}{5}.$$

Thus,

$$\left| \sum_{j=1}^{\infty} x^j_{n_k} y^j \right| \geq \varepsilon - \frac{4\varepsilon}{5} = \frac{\varepsilon}{5}$$

which contradicts the weak convergence of $\{x_{n_k}\}$ to zero. Hence the result. ■

Remark 5.1.1 The space ℓ_1 being infinite dimensional, the weak and norm topologies are different. Nevertheless, we see from the preceding proposition that the convergent sequences for these topologies are the same. While two metric spaces which have the same convergent sequences are equivalent (*i.e.* their topologies are the same), two topological spaces with the same convergent sequences need not be the same. This illustrates the inadequacy of considering just sequences in a general topological space. We also conclude that the weak topology on ℓ_1 is not metrizable. ■

Definition 5.1.3 *Let V and W be Banach spaces and let $T : V \to W$ be a linear mapping. We say that T is* **weakly continuous** *if T is continuous as a mapping from V into W, each space being endowed with its weak topology.* ■

Lemma 5.1.1 *Let $T : V \to W$ be a linear mapping. Then T is weakly continuous if, and only if, for every $f \in W^*$, the map $x \mapsto f(T(x))$ is a weakly continuous map from V into $\mathbb{R}$ (or $\mathbb{C}$, in case of complex Banach spaces).*

Proof: If T is weakly continuous, then, clearly its composition with any $f \in W^*$ will also be weakly continuous.

Now let $f \in W^*$. Let U be open in $\mathbb{R}$ (or $\mathbb{C}$). Then $f^{-1}(U)$ is weakly open in W, by definition. If $f \circ T$ is weakly continuous, we also have that $T^{-1}(f^{-1}(U)$ is weakly open in V. But by the definition of the weak topology, every weakly open set in W is the union of finite intersections of sets of the form $f^{-1}(U)$, where U is open in $\mathbb{R}$ (or $\mathbb{C}$) and $f \in W^*$. Thus, it follows from the above that the inverse image of every weakly open set in W is weakly open in V, *i.e.* T is weakly continuous. ■

Our final result in this section shows that as far as continuity of linear maps is concerned, the topology really does not matter.

Theorem 5.1.1 *Let V and W be Banach spaces and let $T : V \to W$ be a linear map. then $T \in \mathcal{L}(V, W)$ if, and only if, T is weakly continuous.*

Proof: Let $T \in \mathcal{L}(V, W)$. If $f \in W^*$, then the map $f \circ T \in V^*$ and so is weakly continuous as well. Thus by lemma 5.1.1, it follows that T is weakly continuous.

Conversely, if T is weakly continuous, since the weak topology is Hausdorff, it follows that the graph $G(T)$ is closed when $V \times W$ is given the product topology induced by the weak topologies (cf. Lemma 4.4.1). But this is clearly the weak topology of $V \times W$ (why?) and so $G(T)$ is weakly closed in $V \times W$ and so is closed for its norm topology as well. The continuity of T (between the norm topologies of V and W) is now a consequence of the closed graph theorem. ■

5.2 The Weak* Topology

Let V be a Banach space. Then its dual space, V^*, has its natural norm topology. It also is endowed with its weak topology, *viz.* the coarsest topology such that all the elements of V^{**} are continuous. We now define an even coarser topology on V^*.

Definition 5.2.1 *The* **weak* topology** *on V^* is the coarsest topology such that the functionals $\{J_x \mid x \in V\}$ are all continuous, where $J : x \mapsto J_x$ is the canonical imbedding of V into V^{**}.* ■

Clearly, the weak* topology is coarser than the weak topology on V^*. Thus if $\mathcal{S}, \mathcal{W}$ and $\mathcal{W}^*$ denote the norm, weak and weak* topologies, respectively, on V^*, we have

$$\mathcal{W}^* \subset \mathcal{W} \subset \mathcal{S}.$$

Remark 5.2.1 It is clear that if V is a reflexive Banach space, then the weak and weak* topologies on V^* coincide. ■

Proposition 5.2.1 *Let V be a Banach space. The weak* topology on V^* is Hausdorff.*

Proof: Let f_1 and f_2 be distinct elements of V^*. Then, there exists $x \in V$ such that $f_1(x) \neq f_2(x)$. Choose disjoint neighbourhoods U_1 of $f_1(x)$ and U_2 of $f_2(x)$ in $\mathbb{R}$ (or $\mathbb{C}$, as the case may be). Then, by definition, the sets

$$J_x^{-1}(U_1) \; = \; \{f \in V^* \mid f(x) \in U_1\} \text{ and } J_x^{-1}(U_2) \; = \; \{f \in V^* \mid f(x) \in U_2\}$$

are both weak* open sets and are clearly disjoint and contain f_1 and f_2 respectively. This completes the proof. ■

As in the case of the weak topology, we can describe the weak* open neighbourhoods of elements of V^* as follows. Let I be a finite indexing set and let $x_i \in V$ for $i \in I$. Let $\varepsilon > 0$. Then, a weak* open neighbourhood of $f_0 \in V^*$ can be written as

$$\{f \in V^* \mid |(f - f_0)(x_i)| < \varepsilon, \; i \in I\}.$$

Notation: Let $\{f_n\}$ be a sequence in V^*. If f_n converges to f in V^* in the norm topology, we write $f_n \to f$. If it converges to f in the weak topology of V^*, we will write, as before, $f_n \rightharpoonup f$. If the sequence converges in the weak* topology of V^*, we will write $f_n \overset{*}{\rightharpoonup} f$.

The proof of the following proposition is easy and is left to the reader as an exercise.

Proposition 5.2.2 *Let V be a Banach space and let $\{f_n\}$ be a sequence in V^*.*

(i) *$f_n \overset{*}{\rightharpoonup} f$ in V^* if, and only if, $f_n(x) \to f(x)$ for every $x \in V$.*

(ii) $f_n \to f \;\Rightarrow\; f_n \rightharpoonup f \;\Rightarrow\; f_n \overset{*}{\rightharpoonup} f$.

(iii) *If $f_n \overset{*}{\rightharpoonup} f$ in V^* and $x_n \to x$ in V, then $f_n(x_n) \to f(x)$.* ■

The next proposition shows that the functionals $\{J_x \mid x \in V\}$ are the only ones which are continuous with respect to the weak* topology.

Proposition 5.2.3 *Let φ be a linear functional on V^* which is continuous with respect to the weak* topology. Then, there exists $x \in V$ such that $\varphi = J_x$.*

Proof: Let $\widetilde{D}$ be the open unit ball in $\mathbb{R}$ (or $\mathbb{C}$, in case of complex Banach spaces). Since φ is weak* continuous, there exists a weak* neighbourhood of the origin in V^*, say, U, such that $\varphi(U) \subset \widetilde{D}$. Assume that

$$U = \{f \in V^* \mid |f(x_i)| < \varepsilon, \ 1 \le i \le n\}$$

where $\varepsilon > 0$ and $x_i \in V$ for $1 \le i \le n$. Thus, for every $f \in U$, we have that

$$|\varphi(f)| < 1.$$

Assume that for some $f \in V^*$, we have $f(x_i) = 0$ for all $1 \le i \le n$. Then $f \in U$. Further, for any real number k, we have that $kf(x_i) = 0$ for all $1 \le i \le n$ and so $kf \in U$ as well. Thus, for all positive integers k,

$$|\varphi(f)| < \frac{1}{k}$$

and so $\varphi(f) = 0$. It then follows (cf. Exercise 3.17) that there exist scalars α_i for $1 \le i \le n$ such that

$$\varphi = \sum_{i=1}^{n} \alpha_i J_{x_i}.$$

This proves the result with $x = \sum_{i=1}^{n} \alpha_i x_i$. ■

Corollary 5.2.1 *A weak* closed hyperplane must be of the form*

$$H = \{f \in V^* \mid f(x) = \alpha\}$$

where $x \in V$ and α is a scalar.

Proof: For simplicity, we will assume that the base field is $\mathbb{R}$. Since H is a weak* closed hyperplane, it is closed in the norm topology as well and so (cf. Proposition 3.2.1) there exists $\varphi \in V^{**}$ such that

$$H = \{f \in V^* \mid \varphi(f) = \alpha\}$$

for some real number α. Let $f_0 \in H^c$, the complement of H. Since H is weak* closed, there exists a weak* open neighbourhood of the form

$$U = \{f \in V^* \mid |(f - f_0)(x_i)| < \varepsilon, \ 1 \le i \le n\}$$

(where $\varepsilon > 0$ and $x_i \in V$ for $1 \le i \le n$) of f_0 contained in H^c. Now, U is a convex set. Thus, it is easy to see that either $\varphi(f) < \alpha$ for all $f \in U$

or $\varphi(f) > \alpha$ for all $f \in U$. Assume the former (the proof in the latter case will be similar). Let $W = U - \{f_0\} = \{f - f_0 \mid f \in U\}$. Then,

$$W = \{g \in V^* \mid g + f_0 \in U\}$$

and so $g \in W$ if, and only if, $-g \in W$. Thus $W = -W$. Now, if $\varphi(f) < \alpha$ for all $f \in U$, it follows that

$$\varphi(g) \ < \ \alpha - \varphi(f_0)$$

for all $g \in W$. Since $W = -W$, it then follows that

$$|\varphi(g)| \ < \ |\alpha - \varphi(f_0)|$$

for all $g \in W$. Since we can always find $f_0 \in H^c$ such that $|\alpha - \varphi(f_0)| < \eta$, for any $\eta > 0$, and since W is weak* open, it follows that φ is weak* continuous at the origin, and so, by linearity, weak* continuous everywhere. Then, by the preceding proposition, $\varphi = J_x$ for some $x \in V$. This completes the proof. ■

In a finite dimensional space, we have $\dim(V) = \dim(V^*) = \dim(V^{**})$ and so the canonical imbedding $J : V \to V^{**}$ is onto. Thus every finite dimensional space V is reflexive and the weak and weak* topologies on V^* coincide.

However, the above corollary shows that , in infinite dimensional and nonreflexive spaces, the weak* topology is strictly coarser than the weak topology. If $\varphi \in V^{**} \backslash J(V)$, then the hyperplane $[\varphi = \alpha]$ is a convex and (norm) closed set and hence is weakly closed but it is not weak* closed.

One might wonder the purpose of impoverishing the norm topologies on Banach spaces and their duals to produce the weak and weak* topologies. One important off shoot of this is process is that by decreasing the number of open sets, we increase the chances of a set being compact, which is a very useful and topological property. We saw, in Chapter 2, that in infinite dimensional spaces, the closed unit ball cannot be compact. The ball becomes compact in the weak* topology.

Theorem 5.2.1 *(***Banach-Alaoglu Theorem***) Let V be a Banach space. Then, B^*, the closed unit ball in V^*, is weak* compact.*

Proof: Consider the product space

$$X \ = \ \Pi_{x \in V}[-\|x\|, \|x\|]$$

with the usual product topology inherited from $\mathbb{R}$. This space is clearly compact since each bounded and closed interval in $\mathbb{R}$ is compact. Let $f \in B^*$. Then, for each $x \in V$, we have $f(x) \in [-\|x\|, \|x\|]$. Thus, the map $f \mapsto \varphi(f) = (f(x))_{x\in V}$ is a bijection from B^* onto its image in X. If B is endowed with the topology induced by the weak* topology of V^*, then the definitions of this topology and the product topology on X tell us that φ is a homeomorphism. We thus just need to show that $\varphi(B^*)$ is closed in X, which will prove $\varphi(B^*)$, and hence B^*, to be compact.

Let $(f_x)_{x\in V} \in \overline{\varphi(B^*)}$. Define, for $x \in V$, $f(x) = f_x$. The proof will be complete if we can show that f is linear; since $|f(x)| \le \|x\|$ for all $x \in V$, it will then follow that $f \in B^*$, *i.e.* $\overline{\varphi(B^*)} = \varphi(B^*)$.

Let $\varepsilon > 0$. Then, given x and $y \in V$, we can find $g \in B^*$ such that

$$|g(x) - f(x)| \;<\; \frac{\varepsilon}{3},\; |g(y) - f(y)| \;<\; \frac{\varepsilon}{3},\; |g(x+y) - f(x+y)| \;<\; \frac{\varepsilon}{3}.$$

Thus,

$$|f(x+y) - f(x) - f(y)| \;<\; \varepsilon$$

and, since ε was arbitrarily chosen, we deduce that

$$f(x+y) \;=\; f(x) + f(y)$$

for every x and $y \in V$. Similarly, we can show that if α is a scalar and if $x \in V$,

$$f(\alpha x) \;=\; \alpha f(x).$$

Thus f is linear and the proof is complete. ■

Lemma 5.2.1 *Let V be a Banach space and let $f_i \in V^*$, $1 \le i \le n$. Let α_i, $1 \le i \le n$ be scalars. Then, the following are equivalent:*
(i) *For every $\varepsilon > 0$, there exists $x_\varepsilon \in V$ with $\|x_\varepsilon\| \le 1$ and such that*

$$|f_i(x_\varepsilon) - \alpha_i| \;<\; \varepsilon$$

for all $1 \le i \le n$.
(ii) *For all scalars β_i, $1 \le i \le n$, we have*

$$\left|\sum_{i=1}^{n} \beta_i\alpha_i\right| \;\le\; \left\|\sum_{i=1}^{n} \beta_i f_i\right\|.$$

Proof: (i) $\Rightarrow$ (ii). Let $s = \sum_{i=1}^{n} |\beta_i|$. By (i),

$$\left|\sum_{i=1}^{n}(\beta_i f_i(x_\varepsilon) - \beta_i\alpha_i)\right| \;<\; \varepsilon s$$

which implies that

$$\left|\sum_{i=1}^{n} \beta_i \alpha_i\right| \;\leq\; \varepsilon s + \left\|\sum_{i=1}^{n} \beta_i f_i\right\|$$

from which (ii) follows, since we can choose $\varepsilon > 0$ to be arbitrarily small.

(ii) $\Rightarrow$ (i). Let $\overline{\alpha} = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$ (the proof when the scalar field is $\mathbb{C}$ is similar). Define $\mathcal{A} : V \to \mathbb{R}^n$ by $\mathcal{A}(x) = (f_1(x), \cdots, f_n(x))$. We then need to show that $\overline{\alpha} \in \overline{\mathcal{A}(B)}$ where B is the closed unit ball in V. If not, by the Hahn-Banach theorem, we can find scalars λ and $\beta_1, \cdots, \beta_n$ such that, for every $x \in B$,

$$\sum_{i=1}^{n} \alpha_i \beta_i \;>\; \lambda \;>\; \sum_{i=1}^{n} \beta_i f_i(x)$$

which implies that

$$\left\|\sum_{i=1}^{n} \beta_i f_i\right\| \;\leq\; \lambda \;<\; \sum_{i=1}^{n} \alpha_i \beta_i$$

which contradicts (ii). This completes the proof. ■

Proposition 5.2.4 *Let V be a Banach space. Let B be the closed unit ball in V and B^{**} the closed unit ball in V^{**}. Let $J : V \to V^{**}$ be the canonical imbedding. Then, B^{**} is the weak* closure of $J(B)$ in V^{**}.*

Proof: Since B^{**} is weak* compact, it is weak* closed. Let $\varphi_0 \in B^{**}$. Consider a weak* open neighbourhood of φ_0 of the form

$$U \;=\; \{\varphi \in V^{**} \mid |(\varphi - \varphi_0)(f_i)| < \varepsilon,\ 1 \leq i \leq n\}$$

where $\varepsilon > 0$ and $f_i \in V^*$, $1 \leq i \leq n$. Let $\alpha_i = \varphi_0(f_i)$, $1 \leq i \leq n$. Then, for scalars β_i, $1 \leq i \leq n$, we have,

$$\left|\sum_{i=1}^{n} \beta_i \alpha_i\right| \;=\; \left|\varphi_0\left(\sum_{i=1}^{n} \beta_i f_i\right)\right| \;\leq\; \left\|\sum_{i=1}^{n} \beta_i f_i\right\|.$$

Then, by the preceding lemma,there exists $x \in B$ such that $J_x \in U$. Thus U intersects $J(B)$ and this shows that $J(B)$ is weak* dense in B^{**} which completes the proof. ■

Remark 5.2.2 Let V be a Banach space. Since the map $J : V \to V^{**}$ is an isometry, it follows that $J(B)$ is closed in the norm topology of V^{**}. Thus, either $J(B) = B^{**}$, which is true if, and only if, V is reflexive, or $J(B)$ is a closed and proper subset of B^{**}. Thus, in the non-reflexive case, $J(B)$ is not dense in B^{**} for the norm topology. ■

5.3 Reflexive Spaces

Let us recall that a Banach space V is said to be reflexive if the canonical imbedding $J : V \to V^{**}$ is surjective (cf. Definition 3.1.1). We also saw that the spaces ℓ_p for $1 < p < \infty$ are examples of reflexive spaces while ℓ_1 is not reflexive. In Chapter 7, we will see that every *Hilbert space* is reflexive.

In this section, we will study some important properties of reflexive spaces.

Notation: Given a Banach space V, we will denote the closed unit balls in V, V^* and V^{**} by B, B^* and B^{**}, respectively.

Theorem 5.3.1 *A Banach space V is reflexive if, and only if, B is weakly compact.*

Proof: Assume that B is weakly compact. Since $J : V \to V^{**}$ is an isometry, it is continuous and hence weakly continuous as well (cf. Theorem 5.1.1) and so $J(B)$ is weakly compact. Hence it is weak* compact as well. The weak* topology being Hausdorff, it follows that $J(B)$ is weak* closed. But then (cf. Proposition 5.2.4) it follows that $J(B) = B^{**}$. This immediately implies that J is surjective, *i.e.* V is reflexive.

Conversely, let V be reflexive. Then the weak and weak* topologies on V^* coincide. Hence, by the Banach-Alaoglu theorem, B^* is weakly compact. Then, by the preceding arguments, it follows that V^* is reflexive. Then, just as we saw earlier, it follows that B^{**} is weakly compact. Since V is reflexive, we have $B = J^{-1}(B^{**})$. Also, since $J^{-1} : V^{**} \to V$ is continuous, it is weakly continuous as well and so B is weakly compact. ■

Corollary 5.3.1 *Let V and W be Banach spaces and let $T : V \to W$ be an isometric isomorphism. Then, if V is reflexive, so is W.*

Proof: Let B_V and B_W be the closed unit balls in V and W, respectively. Since T is an isometric isomorphism, we have that $T(B_V) = B_W$. Now, T being continuous, it is weakly continuous as well. Since V is reflexive, we have that B_V is weakly compact and so $B_W = T(B_V)$ is also weakly compact, which implies that W is reflexive. ■

Corollary 5.3.2 *Let V be a reflexive Banach space and let W be a* closed *subspace of V. The W is also reflexive.*

Proof: It is easy to see that the weak topology on W is none other than the topology induced on W by the weak topology of V (cf. Exercise 5.1). Since V is reflexive, it follows that B is weakly compact. The unit ball in W is none other than $W \cap B$. But W being a closed subspace, it is weakly closed and since B is weakly compact, it follows that $W \cap B$ is weakly compact as well. Thus, it follows that W is reflexive. ■

Corollary 5.3.3 *Let V be a Banach space. Then, V is reflexive if, and only if, V^* is reflexive.*

Proof: We already saw in the proof of Theorem 5.3.1 that if V is reflexive, then V^* is reflexive.

Conversely, let V^* be reflexive. Then, as before, V^{**} is reflexive. Now, $J(V)$ is a closed subspace of V^{**} and so, by the preceding corollary, it follows that $J(V)$ is reflexive. But then $J^{-1} : J(V) \to V$ is an isometric isomorphism and so V is reflexive by Corollary 5.3.1. ■

Corollary 5.3.4 *Let V be a reflexive Banach space. Let $K \subset V$ be a closed, bounded and* convex *subset. Then, K is weakly compact.*

Proof: Since K is bounded, there exists a positive integer m such that $K \subset mB$. Then, since K is convex and closed, it is weakly closed and since mB is weakly compact, it follows that K is weakly compact. ■

Proposition 5.3.1 *Let V and W be Banach spaces, with W being reflexive, and let $A : D(A) \subset V \to W$ be a linear transformation which is* closed *and* densely defined. *Then A^* is also densely defined.*

Proof: Let $\varphi \in W^{**}$ which vanishes on $D(A^*)$. It suffices to show that $\varphi = \mathbf{0}$. Since W is reflexive, there exists $y \in W$ such that

$$< \varphi, v >_{W^{**},W^*} = < v, y >_{W^*,W}$$

for all $v \in W^*$. Thus, we need to show that $y = \mathbf{0}$ given that $< w, y >_{W^*,W} = 0$ for all $w \in D(A^*)$. If not, then $(\mathbf{0}, y) \notin G(A)$, the graph of A. Since $G(A)$ is closed, by hypothesis, there exists $(f, v) \in V^* \times W^*$ such that

$$< f, u >_{V^*,V} + < v, A(u) >_{W^*,W} = 0 \quad (5.3.1)$$

for all $u \in D(A)$ and such that $< v, y >_{W^*,W} \neq 0$, by virtue of the Hahn-Banach theorem (cf. Corollary 3.2.1). It follows from (5.3.1) that $v \in D(A^*)$ which then implies that $< v, y >_{W^*,W} = 0$ which is a contradiction. ■

Thus, in the circumstances of the above proposition, we can define the second adjoint $A^{**} = (A^*)^*$ from V^{**} into W^{**}. If we now assume that both V and W are reflexive, then we can identify V with V^{**} and W with W^{**} via their respective canonical imbeddings. In this case, we will then have $A^{**} : D(A^{**}) \subset V \to W$.

Theorem 5.3.2 *Let V and W be* reflexive *Banach spaces and let $A : D(A) \subset V \to W$ be a* closed *and* densely defined *linear transformation. Then, $A^{**} = A$.*

Proof: It suffices to show that the graphs $G(A)$ and $G(A^{**})$ are the same. Recall that if we define $\mathcal{J} : W^* \times V^* \to V^* \times W^*$ by $\mathcal{J}(v, f) = (-f, v)$, we then have $\mathcal{J}(G(A^*)) = G(A)^\perp$ (cf. Proposition 4.7.2). Then $(G(A)^\perp)^\perp = (\mathcal{J}(G(A^*))^\perp \subset V \times W$. Thus, $(G(A)^\perp)^\perp$ consists of all $(v, w) \in V \times W$ such that

$$< -A^*(\varphi), v >_{V^*,V} + < \varphi, w >_{W^*,W} = 0$$

for all $\varphi \in D(A^*)$. This is equivalent to saying that $v \in D(A^{**})$ and that $w = A^{**}(v)$. Thus,

$$G(A^{**}) = (G(A)^\perp)^\perp = \overline{G(A)} = G(A)$$

since $G(A)$ is closed and this completes the proof. ■

5.4 Separable Spaces

In this section, we will study the relationship between separable spaces and weak topologies.

Definition 5.4.1 *A topological space is said to be* **separable** *if it contains a countable dense set.* ■

Proposition 5.4.1 *Let V be a Banach space. If V^* is separable, then so is V.*

Proof: Let $\{f_n\}$ be a countable dense set in V^*. Choose $\{x_n\}$ in V such that

$$\|x_n\| = 1 \text{ and } f_n(x_n) > \frac{1}{2}\|f_n\|.$$

Assume, for simplicity, that the base field is $\mathbb{R}$. Let W be the linear subspace generated by the sequence $\{x_n\}$ and let W_0 be the set of all finite linear combinations of the $\{x_n\}$ with *rational* coefficients. Then W_0 is countable and it is dense in W. So it suffices to show that W is dense in V. Let $f \in V^*$ which vanishes on W. We need to show then that f vanishes on all of V (*i.e.* f is identically zero). Let $\varepsilon > 0$. Then, there exists f_m such that $\|f - f_m\| < \varepsilon$, by the density of the $\{f_n\}$. Since $f(x_n) = 0$ for all n, we have

$$\frac{1}{2}\|f_m\| < f_m(x_m) = (f_m - f)(x_m) \leq \|f_m - f\|.$$

Thus,

$$\|f\| \leq \|f_m - f\| + \|f_m\| < 3\varepsilon$$

from which it follows that $f = \mathbf{0}$ since ε was arbitrarily chosen. ■

Example 5.4.1 The converse of the above proposition is not true. We know that $\ell_1^* = \ell_\infty$. While ℓ_1 is separable (the set of all sequences with only a finite number of non-zero components, all of which are rational, forms a countable dense set of ℓ_1), ℓ_∞ is not separable. To see this, we prove that no countable set in ℓ_∞ can be dense. Indeed, let $\{f_n\}$ be a countable set in ℓ_∞ where $f_n = (f_n^i)$. Define $f = (f^i)$ by

$$f^i = \begin{cases} 0, & \text{if } |f_i^i| \geq 1, \\ 2, & \text{if } |f_i^i| < 1. \end{cases}$$

Then $f \in \ell_\infty$ and $\|f - f_n\|_\infty \geq 1$ for all n. Thus $\{f_n\}$ cannot be dense in ℓ_∞. ■

Corollary 5.4.1 *Let V be a Banach space. Then V is both separable and reflexive if, and only if, V^* is both separable and reflexive.*

Proof: If V^* is both separable and reflexive, then so is V. Conversely, if V is separable and reflexive, so is $J(V) = V^{**}$, where J is the canonical imbedding of V into V^{**}. Thus, it now follows that V^* is separable and reflexive. ■

Theorem 5.4.1 *Let V be a Banach space. Then, V is separable if, and only if, the weak* topology on B^*, the closed unit ball in V^*, is metrisable.*

Proof: Assume that V is separable. Let $\{x_n\}$ be a countable dense set in V. We may assume, without loss of generality, that $x_n \neq \mathbf{0}$ for all n (why?). For f and $g \in B^*$, define

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n \|x_n\|} |(f-g)(x_n)|. \qquad (5.4.1)$$

It is easy to check that $d(.,.)$ is well defined and that it defines a metric on B^*. Let U be a weak* open neighbourhood of $f_0 \in B^*$ of the form

$$U = \{f \in B^* \mid |(f-f_0)(y_i)| < \varepsilon,\ 1 \leq i \leq k\}$$

where $\varepsilon > 0$ and $y_i \in V$ for $1 \leq i \leq k$. Since $\{x_n\}$ is dense, there exists x_{n_i} such that $\|y_i - x_{n_i}\| < \varepsilon/4$ for each $1 \leq i \leq k$. Now choose $r > 0$ such that

$$r 2^{n_i} \|x_{n_i}\| < \frac{\varepsilon}{2} \text{ for all } 1 \leq i \leq k.$$

Consider the ball $B_d(f_0; r)$ in B^* provided with the metric defined in (5.4.1). If f belongs to this ball, *i.e.* $d(f, f_0) < r$, then for each $1 \leq i \leq k$, we have

$$\begin{array}{rcl} |(f-f_0)(y_i)| & \leq & |(f-f_0)(y_i - x_{n_i})| + |(f-f_0)(x_{n_i})| \\ & \leq & 2.\frac{\varepsilon}{4} + 2^{n_i}\|x_{n_i}\| r \\ & < & \varepsilon. \end{array}$$

Thus $B_d(f_0; r) \subset U$ and so every weak* open set is also open in the metric topology.

On the other hand, consider a ball $B_d(f_0; r)$. Consider the weak* open neighbourhood of f_0 given by

$$U_k^\varepsilon = \left\{ f \in B^* \mid \left| (f-f_0)\left(\frac{1}{\|x_i\|} x_i\right) \right| < \varepsilon,\ 1 \leq i \leq k \right\}.$$

Choose $\varepsilon < r/2$ and k such that

$$\sum_{n=k+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^k} < \frac{r}{4}.$$

If $f \in U_k^\varepsilon$, then

$$\begin{array}{rcl} d(f,f_0) & = & \sum_{n=1}^k \frac{1}{2^n\|x_n\|}|(f-f_0)(x_n)| + \sum_{n=k+1}^\infty \frac{1}{2^n\|x_n\|}|(f-f_0)(x_n)| \\ & < & \varepsilon\sum_{n=1}^k \frac{1}{2^n} + 2\sum_{n=k+1}^\infty \frac{1}{2^n} \\ & < & \frac{r}{2} + 2\frac{r}{4} \\ & = & r. \end{array}$$

Thus, $U_k^\varepsilon \subset B_d(f_0;r)$ which shows that every open set in the metric topology is also weak* open. Thus, the weak* and the metric topologies on B^* are the same.

Conversely, assume that the weak* topology on B^* is metrisable. Consider, for each positive integer n, the ball $B_d(\mathbf{0};\frac{1}{n})$, where d is the metric defined on B^*. This ball then contains a weak* open neighbourhood of zero, say U_n which can be written in the form

$$U_n = \{f \in B^* \mid |f(x)| < \varepsilon_n, \text{ for all } x \in \Phi_n\}$$

where $\varepsilon_n > 0$ and Φ_n is a finite set in V. The set

$$D = \cup_{n=1}^\infty \Phi_n$$

is then countable and the set E of all finite rational linear combinations of the elements of D is a countable set which will be dense in the subspace generated by D. If $f \in V^*$ is such that $f(x) = 0$ for all x in the subspace generated by D, then clearly, $f \in U_n$ for each n. Thus,

$$f \in \cap_{n=1}^\infty U_n \subset \cap_{n=1}^\infty B_d(\mathbf{0};1/n) = \{\mathbf{0}\}.$$

Thus the subspace generated by D is itself dense in V and so the countable set E is also dense in V and hence V is separable.

This completes the proof. ■

Corollary 5.4.2 *Let V be a separable Banach space. Then, every bounded sequence in V^* has a weak* convergent subsequence.*

Proof: A bounded sequence in V^* is contained in some ball, which is weak* compact. Since V is separable, the weak* topology on this ball is metrisable and so the ball is weak* sequentially compact as well. ■

In a metric space compactness and sequential compactness are equivalent (cf. Proposition 1.2.6); this is not true in a general topological space. Thus a sequence in a compact topological space may fail to have a convergent subsequence, as the following example shows.

Example 5.4.2 Consider the space ℓ_∞. Define

$$f_n(x) = x_n$$

for $x = (x_n) \in \ell_\infty$. Then clearly $f_n \in \ell_\infty^*$ and, further, $\|f_n\|_{\ell_\infty^*} = 1$, for all n. The unit ball in ℓ_∞^* is weak* compact, by the Banach-Alaoglu theorem. Assume, if possible, that there exists a weak* convergent subsequence $\{f_{n_k}\}$ for this sequence. This, in view of Proposition 5.2.2 (i), imples that $\{x_{n_k}\}$ is convergent for every $x = (x_n) \in \ell_\infty$, which is clearly absurd. Thus $\{f_n\}$ cannot have any weak* convergent subsequence, eventhough it lies in a weak* compact set. ■

Example 5.4.3 We know that (cf. Exercise 3.7) $c_0^* = \ell_1$ and that $\ell_1^* = \ell_\infty$. Consider the sequence $\{\mathsf{e}_n\}$ in ℓ_1, where e_n is the sequence whose n-th entry is unity, all other entries being zero. If this sequence has a weakly convergent subsequence, then the weak limit has to be zero. To see this, fix a positive integer l. Then $\mathsf{e}_l \in \ell_\infty$ and so if we have a subsequence $\{\mathsf{e}_{n_k}\}$ weakly converging to $x \in \ell_1$, it follows that

$$< \mathsf{e}_l, x >_{\ell_\infty,\ell_1} = \lim_{k\to\infty} < \mathsf{e}_l, \mathsf{e}_{n_k} >_{\ell_\infty,\ell_1} = 0.$$

Thus $x_l = 0$ for all l, *i.e.* $x = \mathbf{0}$. But if $\{\mathsf{e}_{n_k}\}$ converges weakly in ℓ_1, it also converges in norm by Schur's lemma (cf. Proposition 5.1.5) and so this subsequence should converge in norm to zero which is impossible, since $\|\mathsf{e}_n\|_1 = 1$ for all n. Thus $\{\mathsf{e}_n\}$ has no weakly convergent subsequence in ℓ_1. But since ℓ_1 is separable and so c_0 is also separable (cf. Proposition 5.4.1) and, by the preceding corollary, $\{\mathsf{e}_n\}$ must have a weak* convergent subsequence. In fact, if $x = (x_n) \in c_0$, we have

$$< \mathsf{e}_n, x >_{\ell_1,c_0} = x_n$$

which converges to zero. Thus $\{\mathsf{e}_n\}$ weak* converges to zero. Thus the weak and weak* convergent sequences in ℓ_1 are not the same (while its norm and weakly convergent sequences are the same). ■

Theorem 5.4.2 *Let V be a reflexive Banach space. Then every bounded sequence has a weakly convergent subsequence.*

Proof: Let $\{x_n\}$ be a bounded sequence in V. Let $W = \overline{\text{span}\{\{x_n\}\}}$, *i.e.* the smallest closed subspace containing the sequence in question. Then, W is also reflexive and, by construction, it is separable (why?). So W^* is also reflexive and separable. Then, every bounded sequence in W^{**} has a weak* convergent subsequence and since W^{**} is also reflexive, the weak and weak* topologies are the same. In particular, $\{J(x_n)\}$ has a weakly convergent subsequence in W^{**}, where $J : W \to W^{**}$ is the canonical imbedding, and since $J^{-1} : W^{**} \to W$ is an isometry and hence weakly continuous, $\{x_n\}$ has a weakly convergent subsequence in W. Since the weak topology in W is the topology induced on W by the weak topology on V (cf. Exercise 5.1), it follows that this subsequence converges weakly in V as well. ■

Remark 5.4.1 The converse of the above theorem is also true and it is a deep result due to Eberlein and Šmulian: if every bounded sequence admits a weakly convergent subsequence in a Banach space, then the space is reflexive. ■

5.5 Uniformly Convex Spaces

We know that the unit ball in a normed linear space is convex. However, the nature of the boundary of this ball depends on the norm. For instance, in $\mathbb{R}^2$, with the euclidean metric (*i.e.* $\mathbb{R}^2 = \ell_2^2$), the unit ball is a very symmetric object which 'bulges uniformly' in all directions. On the other hand, if we consider $\mathbb{R}^2$ as ℓ_1^2 or as ℓ_∞^2, then the unit ball will be, the rhombus bounded by the lines $(\pm x_1) + (\pm x_2) = 1$ or the unit square, respectively. In both these cases, the boundary has a lot of 'flat' portions. Uniform convexity makes precise the notion of the boundary 'bulging uniformly' in all directions. This is a condition describing the 'geometry' of the norm, but has an important 'analytic' consequence, which will be the main theorem of this section. It also has important consequences in the calculus of variations, which we will see in the next section.

Definition 5.5.1 *A normed linear space is said to be* **uniformly convex** *if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever we have x and $y \in V$ satisfying*

$$\|x\| \ \leq \ 1, \ \|y\| \ \leq \ 1 \text{ and } \|x - y\| \ > \ \varepsilon,$$

it follows that

$$\left\|\frac{1}{2}(x+y)\right\| \;<\; 1-\delta. \blacksquare$$

In other words, given two points on the boundary which are at a distance of ε from each other, then, irrespective of the position of these points, the mid-point of the chord joining them should lie in the interior, at a minimum distance away from the boundary, the minimum distance being prescribed uniformly.

Uniform convexity is stronger than the notion of strict convexity (cf. Exercise 3.2).

Example 5.5.1 The spaces ℓ_1^N and ℓ_∞^N are not uniformly convex. In fact, they are not even strictly convex. ■

Example 5.5.2 The space ℓ_2^N is uniformly convex. Let x and $y \in \ell_2^N$. Then it is easy to verify that

$$\left\|\frac{1}{2}(x+y)\right\|_2^2 + \left\|\frac{1}{2}(x-y)\right\|_2^2 \;=\; \frac{1}{2}(\|x\|_2^2+\|y\|_2^2). \tag{5.5.1}$$

If $\|x\|_2 \leq 1,\ \|y\|_2 \leq 1$ and $\|x-y\|_2 > \varepsilon$, with ε sufficiently small, we see that

$$\left\|\frac{1}{2}(x+y)\right\|_2^2 \;<\; 1-\frac{\varepsilon^2}{4} \;=\; (1-\delta)^2$$

where

$$\delta \;=\; 1-\sqrt{1-\frac{\varepsilon^2}{4}}. \blacksquare$$

Remark 5.5.1 When $N = 2$, the relation (5.5.1) is the familiar parallellogram law or Apollonius' theorem in plane geometry. The relation (5.5.1) is also valid for the space ℓ_2 and so ℓ_2 is also uniformly convex. In fact, we will see, in Chapter 7, that this relation is valid in any *Hilbert space* and so every Hilbert space will be uniformly convex. ■

Remark 5.5.2 We will see in Chapter 6, that a relation similar to (5.5.1) is also valid for the spaces ℓ_p^N and ℓ_p whenever $2 \leq p < \infty$ (cf. Proposition 6.2.1) and so all these spaces are uniformly convex. A similar inequality also holds for $1 < p < 2$ but the proof is more difficult. Thus, all these spaces are uniformly convex. ■

Theorem 5.5.1 *Let V be a uniformly convex Banach space. Then V is reflexive.*

Proof: If $J : V \to V^{**}$ is the canonical mapping, it is enough to show that the image of the closed unit ball B in V under J is the closed unit ball B^{**} in V^{**}. Since V is a Banach space, $J(B)$ is a closed set in V^{**} and so it suffices to show that it is dense in B^{**} (cf. Remark 5.2.2).

Let $\varphi \in B^{**}$. Assume that $\|\varphi\|_{V^{**}} = 1$. Let $\varepsilon > 0$. We will show that there exists $x \in B$ such that

$$\|\varphi - J(x)\|_{V^{**}} < \varepsilon.$$

The same will then be true for all elements of B^{**} as well (why?).

Let $\delta > 0$ correspond to ε in the definition of uniform convexity. Choose $f \in V^*$, with $\|f\|_{V^*} = 1$ and such that

$$\varphi(f) \ > \ 1 - \frac{\delta}{2}. \tag{5.5.2}$$

Define

$$U \ = \ \{\xi \in V^{**} \mid |(\xi - \varphi)(f)| < \delta/2\}.$$

Then U is a weak* open neighbourhood of φ in V^{**}. Since $J(B)$ is weak* dense in B^{**} (cf. Proposition 5.2.4), it follows that there exists $x \in B$ such that $J(x) \in U$.

Assume now that $\|J(x) - \varphi\|_{V^{**}} > \varepsilon$. In other words, $\varphi \notin J(x) + \varepsilon B^{**}$. Since εB^{**} is weak* compact (by the Banach-Alaoglu theorem), it is weak* closed and so is its translation by $J(x)$. Thus, there exists a weak* open neighbourhood U_1 of φ such that for all $\xi \in U_1$, we still have $\|\xi - J(x)\|_{V^{**}} > \varepsilon$. Again, as before, there exists $x_1 \in B$ such that $J(x_1) \in U \cap U_1$ by the weak* density of $J(B)$ in B^{**}. Thus,

$$\begin{array}{rcl} |\varphi(f) - f(x)| & < & \frac{\delta}{2} \\ |\varphi(f) - f(x_1)| & < & \frac{\delta}{2} \end{array}$$

and so

$$2\varphi(f) \ < \ \delta + |f(x + x_1)| \ < \ \delta + \|x + x_1\|_V.$$

By virtue of (5.5.2), it follows from the above that

$$\left\|\frac{1}{2}(x + x_1)\right\|_V \ > \ 1 - \delta$$

which contradicts the uniform convexity since $J(x_1) \in U_1$ and so

$$\|x - x_1\|_V \;=\; \|J(x) - J(x_1)\|_{V^{**}} \;>\; \varepsilon$$

while $\|x\|_V = \|x_1\|_V \leq 1$. Thus, it follows that $\|J(x) - \varphi\|_{V^{**}} \leq \varepsilon$ which shows that $J(B)$ is dense in B^{**} as already observed. ■

Remark 5.5.3 The converse of this theorem is not true. A reflexive space need not be uniformly convex. For instance, ℓ_1^N is not uniformly convex, but since it is finite dimensional, it is reflexive. ■

Proposition 5.5.1 *Let V be a uniformly convex Banach space. Let $x_n \rightharpoonup x$ in V. Assume that*

$$\limsup_{n\to\infty} \|x_n\| \;\leq\; \|x\|. \tag{5.5.3}$$

Then $x_n \to x$ in V.

Proof: We already know that (cf. Proposition 5.1.2 (iii))

$$\liminf_{n\to\infty} \|x_n\| \;\geq\; \|x\|. \tag{5.5.4}$$

Thus, by (5.5.3) and (5.5.4), we deduce that $\|x_n\| \to \|x\|$. If $x = \mathbf{0}$, this completes the proof. Assume now that $x \neq \mathbf{0}$. Then, from the convergence of the norms, we deduce that (for large n), $x_n \neq \mathbf{0}$. Set $y_n = x_n/\|x_n\|$ and $y = x/\|x\|$. Observe then that, by hypothesis and the convergence of the norms of x_n, it follows that $y_n \rightharpoonup y$ in V. The proof will be complete if we show that $y_n \to y$.

Since $y_n \rightharpoonup y$, we have

$$1 \;=\; \|y\| \;\leq\; \liminf_{n\to\infty}\left\|\frac{1}{2}(y_n + y)\right\| \;\leq\; \limsup_{n\to\infty}\left\|\frac{1}{2}(y_n + y)\right\| \;\leq\; 1.$$

Thus we have

$$\|y_n\| \;=\; \|y\| \;=\; 1 \text{ and } \left\|\frac{1}{2}(y_n + y)\right\| \;\to\; 1.$$

Hence, by uniform convexity, if $\varepsilon > 0$ is an arbitrary number, we must have $\|y_n - y\| \leq \varepsilon$ for n sufficiently large. This proves that $y_n \to y$ and hence that $x_n \to x$ in V as already observed. ■

5.6 Application: Calculus of Variations

In this section, we will apply the results of the preceding sections to obtain some important results in the calculus of variations, which can be described as the theory of optimization in infinite dimensional spaces.

Proposition 5.6.1 *Let V be a reflexive Banach space and let $K \subset V$ be a non-empty, closed and convex subset. Let $\varphi : K \to \mathbb{R}$ be a convex and lower semi-continuous function. Assume further that*

$$\lim_{\|x\|\to\infty} \varphi(x) = +\infty. \tag{5.6.1}$$

Then, φ attains a minimum in K.

Proof: Fix $x_0 \in K$ and let $\lambda_0 = \varphi(x_0) < \infty$. Set

$$\widetilde{K} = \{x \in K \mid \varphi(x) \leq \lambda_0\}.$$

Since φ is lower semi-continuous and convex, it follows that $\widetilde{K}$ is closed and convex. Further, by (5.6.1), it follows that $\widetilde{K}$ is bounded as well. Thus, $\widetilde{K}$ is weakly closed (cf. Proposition 5.1.4). Now, let $\{x_n\}$ be a minimizing sequence in $\widetilde{K}$ for φ, *i.e.*, $\varphi(x_n) \to \inf_{x\in\widetilde{K}} \varphi(x)$. Then, since the sequence is bounded and since V is reflexive, it has a weakly convergent subsequence, say, $\{x_{n_k}\}$, converging weakly to some $x \in V$. But since $\widetilde{K}$ is weakly closed, we have $x \in \widetilde{K}$. Further, by the lower semi-continuity and convexity of φ, it follows that φ is also weakly lower semi-continuous (cf. Corollary 5.1.1) and so

$$\inf_{y\in\widetilde{K}} \varphi(y) \leq \varphi(x) \leq \liminf_{k\to\infty} \varphi(x_{n_k}) = \inf_{y\in\widetilde{K}} \varphi(y).$$

Thus,

$$\varphi(x) = \min_{y\in\widetilde{K}} \varphi(y).$$

If $z \in K\backslash\widetilde{K}$, then $\varphi(z) > \lambda_0 \geq \varphi(x)$ so that

$$\varphi(x) = \min_{y\in K} \varphi(y).$$

This completes the proof. ■

Remark 5.6.1 The condition (5.6.1) is usually called the condition of *coercivity* of the function(al) φ. Thus, a coercive, convex and lower semi-continuous functional defined on a non-empty, closed and convex subset of a reflexive Banch space always attains a minimum. The method of proof used above is usually known as the *direct method of the calculus of variations.* A minimizing sequence is shown to have a convergent subsequence (in a suitable topology) and the limit is shown to be the desired minimum. ■

Remark 5.6.2 In the proof of the preceding proposition, the coercivity condition was really needed only when K was not bounded.

In a metric space, the lower semi-continuity of a function φ is equivalent to the condition that if $x_n \to x$, then

$$\varphi(x) \leq \liminf_{n\to\infty} \varphi(x_n). \tag{5.6.2}$$

However, in a general topological space, the lower semi-continuity implies the above relation but the converse is not true. A function which satisfies the above relation for all convergent sequences is called *sequentially lower semi-continuous.* In the context of the weak topology, we can thus say that $\varphi : V \to \mathbb{R}$ is *weakly sequentially lower semi-continuous* if whenever $x_n \rightharpoonup x$ in V, we have that (5.6.2) is true. Thus, the preceding proposition holds even when φ is only weakly sequentially lower semi-continuous, since the coercivity condition would imply that every minimizing sequence is bounded and the rest of the proof follows as before. ■

The following result is an immediate consequence of the preceding proposition.

Theorem 5.6.1 *Let V be a reflexive Banach space and let K be a closed convex subset of V. Then, for any $x \in V$, there exists $y \in K$ such that*

$$\|x - y\| = \min_{z\in K} \|x - z\|. \tag{5.6.3}$$

Further, if V is also uniformly convex, then such a y is unique.

Proof: The functional $z \mapsto \|x - z\|$ is clearly coercive, convex and weakly lower semi-continuous. Thus the existence of y follows from the preceding proposition. Assume that V is uniformly convex. Assume that there exist $y_i \in K,\ i = 1, 2$ such that

$$\alpha = \|x - y_1\| = \|x - y_2\| = \min_{z\in K} \|x - z\|.$$

Let us assume that $\|y_1 - y_2\| > \varepsilon > 0$. Then

$$\|(x - y_1) - (x - y_2)\| \; > \; \varepsilon.$$

Then, by the uniform convexity, we have that

$$\left\| x - \frac{1}{2}(y_1 + y_2) \right\| \; = \; \left\| \frac{1}{2}[(x - y_1) + (x - y_2)] \right\| \; < \; \alpha(1 - \delta) \; < \; \alpha$$

for some $\delta > 0$. Since K is convex, we have $\frac{1}{2}(y_1 + y_2) \in K$ and so the above relation contradicts the minimality of y_1 and y_2. Thus it follows that $y_1 = y_2$ and the proof of the uniqueness is complete. ■

Example 5.6.1 In general we have non-uniqueness of the minimizer for the problem (5.6.3) if the space is not uniformly convex. Consider the space ℓ_1^2 (which is $\mathbb{R}^2$ with the norm $\|.\|_1$). It is reflexive since it is finite dimensional, but it is not uniformly convex. Consider the set $K = B$, the closed unit ball in ℓ_1^2, which is a closed and convex set. Let $x = (1, 1)$. Let $y = (a, b) \in K$. Then,

$$\begin{aligned} \|x - y\|_1 &= |1 - a| + |1 - b| \\ &\geq 1 - |a| + 1 - |b| \\ &= 2 - (|a| + |b|) \\ &\geq 1. \end{aligned}$$

However, if $a + b = 1, \; a \geq 0, \; b \geq 0$, then, for all such points $y = (a, b)$, we have

$$\|x - y\|_1 \; = \; 1 - a + 1 - b \; = \; 1.$$

Thus we have uncountably many y which satisfy (5.6.3). ■

If V were not reflexive, then we can guarantee neither the existence, nor the uniqueness.

5.7 Exercises

5.1 Let V be a Banach space and let W be a closed subspace of V. Show that the weak topology on W is the topology induced on W by the weak topology on V.

5.2 Let V be a Banach space and let W be a subspace of V. Show that the closure of W under the weak topology coincides with $\overline{W}$, the closure of W in the norm topology.

5.3 Let V be a Banach space and let W be a subspace of V. Show that $W^\perp \subset V^*$ is weak* closed.

5.4 Let V be a Banach space. Show that V with its weak topology and V^* with its weak* topology are both locally convex topological vector spaces (cf. Remark 3.2.2).

5.5 Use the preceding exercise and Remark 3.2.2 to show that if V is a Banach space and if W is a subspace of V^*, then the weak* closure of W is $(W^\perp)^\perp$. (Compare this with Exercise 3.9.)

5.6 Let V be a Banach space.
(a) Show that $x_n \rightharpoonup x$ in V if, and only if,
(i) $\{\|x_n\|\}$ is bounded and (ii) $f(x_n) \to f(x)$ for all $f \in \mathcal{S}$, where $\mathcal{S}$ is a subset of V^* whose span is dense in V^*.
(b) Show that $f_n \overset{*}{\rightharpoonup} f$ in V^* if, and only if,
(i) $\{\|f_n\|\}$ is bounded and (ii) $f_n(x) \to f(x)$ for all $x \in \mathcal{S}$, where $\mathcal{S}$ is a subset of V whose span is dense in V.

5.7 Let $1 < p < \infty$. Let $x_n = (x_n^j)$ and $x = (x^j)$ be elements of ℓ_p. Show that $x_n \rightharpoonup x$ in ℓ_p if, and only if, $x_n^j \to x^j$ for every positive integer j.

5.8 Let V and W be Banach spaces and let $T : V \to W$ be a linear map. Show that the following are equivalent:
(i) If $x_n \to x$ in V, then $T(x_n) \to T(x)$ in W.
(ii) If $x_n \rightharpoonup x$ in V, then $T(x_n) \rightharpoonup T(x)$ in W.
(iii) If $x_n \to x$ in V, then $T(x_n) \rightharpoonup T(x)$ in W.

5.9 Let V and W be Banach spaces. Let one of them be provided with the norm topology and the other with the weak topology. Let $T : V \to W$ be a linear map. Show that T is continuous if, and only if, $T \in \mathcal{L}(V, W)$.

5.10 Define $T : c_0 \to \ell_1$ as follows: let $x = (x_n) \in c_0$ and

$$T(x) = \left(\frac{x_n}{n^2}\right).$$

Show that $T \in \mathcal{L}(c_0, \ell_1)$. If B is the closed unit ball in c_0, show that

$T(B)$ is not closed in ℓ_1.

5.11 Let V be a reflexive Banach space. If W is a Banach space and if $T \in \mathcal{L}(V, W)$, show that $T(B)$ is closed in W, where B is the closed unit ball in V.

5.12 Show that ℓ_1 does not contain an infinite dimensional subspace that is reflexive.

5.13 Let $1 \leq p < \infty$. Show that ℓ_p is separable.

5.14 Show that $\mathcal{C}[0,1]$ is separable.

5.15 Let $V = \mathcal{C}[0,1]$ and let $K \subset V$ be defined by

$$K = \left\{ f \in V \mid \int_0^{\frac{1}{2}} f(t)\, dt - \int_{\frac{1}{2}}^1 f(t)\, dt = 1 \right\}.$$

(a) Show that K is a closed and convex subset of V.
(b) Show that

$$\inf_{f \in K} \|f\| = 1.$$

(c) Show that K does not admit an element with minimal norm.
(d) Deduce that $\mathcal{C}[0,1]$ is not reflexive.

5.16 Let V be a reflexive real Banach space and let $a(.,.) : V \times V \to \mathbb{R}$ be a continuous bilinear form (cf. Example 4.7.4). Assume that $a(.,.)$ is V-*elliptic* (or, *coercive*), *i.e.* there exists $\alpha > 0$ such that, for all $x \in V$,

$$a(x,x) \geq \alpha \|x\|^2.$$

Let $x \in V$. Define $A(x) : V \to \mathbb{R}$ by

$$A(x)(y) = a(x,y).$$

(a) Show that $A(x) \in V^*$ for every $x \in V$.
(b) Show that $A \in \mathcal{L}(V, V^*)$.
(c) Show that for every $x \in V$,

$$\|A(x)\| \geq \alpha \|x\|.$$

(d) Show that $A : V \to V^*$ is surjective.
(e) Deduce that, for every $f \in V^*$, there exists a unique $x \in V$ such that

$$a(x,y) \;=\; f(y) \tag{5.7.1}$$

for all $y \in V$.

5.17 In the preceding exercise, assume further that $a(.,.)$ is symmetric *i.e.* $a(x,y) = a(y,x)$ for all x and $y \in V$. Let $f \in V^*$. Define, for $x \in V$,

$$J(x) \;=\; \frac{1}{2}a(x,x) - f(x).$$

(a) For any closed convex subset $K \subset V$, show that there exists $x \in K$ such that

$$J(x) \;=\; \min_{y \in K} J(y). \tag{5.7.2}$$

(b) Show that $x \in K$ satisfies (5.7.2) if, and only if,

$$a(x, y-x) \;\geq\; f(y-x) \tag{5.7.3}$$

for every $y \in K$.
(c) Show that the solution $x \in K$ of (5.7.3) (and hence, that of (5.7.2)) is unique.
(d) If K is a closed convex cone (cf. Definition 3.4.1), show that the solution $x \in K$ of (5.7.3) is characterized by

$$a(x,x) \;=\; f(x) \text{ and } a(x,y) \;\geq\; f(y)$$

for all $y \in K$.
(e) If $K = V$, show that the solution x of (5.7.3) is the solution of (5.7.1).